A spherical capacitor consists of two conducting shells of radii 1 cm and 5 cm, held at a potential difference of 120 V. The capacitor is filled with two different dielectrics. The inner dielectric is a shell of silicon dioxide \((k_1 = 3.9)\). This shell has an inner radius of 1 cm and an outer radius of 4 cm. The outer dielectric is a shell of Pyrex glass \((k_2 = 4.6)\), of inner radius 4 cm and outer radius 5 cm.

A) What is the capacitance of this spherical capacitor?

B) What are the total charges on the two plates?

Solution:

A): Let us call the charge on the inner plate \(Q\) and outer plate \(-Q\).

\[ C = \frac{Q}{\Delta V} \]

We need an expression for \(\Delta V\), using \(\Delta V = -\int \mathbf{E} \cdot d\mathbf{x}\). In this case, we first find \(\mathbf{E}\) using Gauss's Law:

\[ \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{enc}}{\epsilon_0} \]

Since we are dealing with dielectrics, the approach will be to first find the field in the absence of vacuum using Gauss's Law, and then divide by \(k\) in the respective region to find the field in the presence of the dielectric. To respect the symmetry of the problem, we choose a spherical Gaussian surface. Since we want the field in between the two plates (so we can integrate from plate to plate), the Guassian surface will be a sphere enclosing only the inner shell. Furthermore, \(\mathbf{E}\) points radially due to the radial symmetry, and since the surface normal also points radially, \(\mathbf{E}\) and \(d\mathbf{A}\) are parallel, so \(\mathbf{E} \cdot \mathbf{A} = EdA\). So we have:

\[ \int EdA = \frac{Q_{enc}}{\epsilon_0} \]
\[ E \int dA = \frac{Q_{enc}}{\epsilon_0} \]
\[ EA = \frac{Q_{enc}}{\epsilon_0} \]
\[ E = \frac{Q}{4\pi r^2 \epsilon_0} \]

So the field in the silicon dioxide and Pyrex glass are, respectively, \(E_1 = \frac{Q}{4\pi r^2 \epsilon_0 k_1}\) and \(E_2 = \frac{Q}{4\pi r^2 \epsilon_0 k_2}\).
Using this, we can find the voltage difference from inner plate to outer plate:

\[ \Delta V = - \int_{r_0}^{r_1} E_1 - \int_{r_1}^{r_2} E_1 \]

\[ = \frac{Q}{4\pi\varepsilon_0} \left[ \frac{1}{k_1} \left( \frac{1}{r_1} - \frac{1}{r_0} \right) + \frac{1}{k_2} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \right] \]

Dividing this into Q, and then plugging in the numbers gives (only absolute value is necessary for capacitance):

\[ C = \text{5.47} \text{ pF} \]

\[ Q = C \Delta V \]
\[ Q = 65.6 \text{ nC} \]

B): The charge for a voltage difference of 120 V is:

\[ Q = C \Delta V \]
\[ Q = 65.6 \text{ nC} \]

This is the charge on the inner shell, and the outer shell is the negative of this. Note that if you carried out the same calculation assuming there were no dielectrics, the capacitance and charge would be lower: dielectrics increase capacitance and therefore stored charge per Volt.